

Hurwitz-type matrices of doubly infinite series*

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This paper show that two doubly infinite series generate a totally nonnegative Hurwitz-type matrix if and only if their ratio represents an \mathcal{S} -functions of a certain kind. The doubly infinite case needs a specific approach, since the ratios have no correspondent Stieltjes continued fraction. Another forthcoming publication (see Dyachenko, arXiv:1608.04440) offers a shorter improved version of this result as well as its application to the Hurwitz stability. Nevertheless, the proof presented here illustrates features of totally nonnegative Hurwitz-type matrices better.

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1 Introduction

This paper offers a weaker version of the main result of the publication [Dy2016]. One of its features (and its key difference from [Dy2016]) is that it tries to make total nonnegativity the cornerstone. Accordingly, some well-known properties are rederived directly from estimates of matrix minors, so the proofs turn to be more self-contained.

Definition. A doubly (*i.e.* two-way) infinite sequence $(f_n)_{n=-\infty}^{\infty}$ is called *totally positive* if all minors of the (four-way infinite) Toeplitz matrix

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & f_0 & f_1 & f_2 & f_3 & f_4 & \dots \\ \dots & f_{-1} & f_0 & f_1 & f_2 & f_3 & \dots \\ \dots & f_{-2} & f_{-1} & f_0 & f_1 & f_2 & \dots \\ \dots & f_{-3} & f_{-2} & f_{-1} & f_0 & f_1 & \dots \\ \dots & f_{-4} & f_{-3} & f_{-2} & f_{-1} & f_0 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} =: T(f), \quad \text{where} \quad f(z) := \sum_{n=-\infty}^{\infty} f_n z^n$$

are nonnegative (*i.e.* the matrix is *totally nonnegative*).

Theorem 1 (Edrei [Edr53]¹). *Let a non-trivial sequence $(f_n)_{n=-\infty}^{\infty}$ be totally positive. Then, unless $f_n = f_0^{1-n} f_1^n$*

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¹Another proof is given in [Kar68, Section 8]; an earlier publication [AESW51] studies the singly infinite case.

for every $n \in \mathbb{Z}$, the series $f(z)$ converges in some annulus to a function with the following representation

$$Cz^j e^{Az + \frac{A_0}{z}} \cdot \frac{\prod_{\mu>0} \left(1 + \frac{z}{\beta_\mu}\right)}{\prod_{\nu>0} \left(1 - \frac{z}{\delta_\nu}\right)} \cdot \frac{\prod_{\mu<0} \left(1 + \frac{z^{-1}}{\beta_\mu}\right)}{\prod_{\nu>0} \left(1 - \frac{z^{-1}}{\delta_\nu}\right)}, \quad (1)$$

where the products converge absolutely, j is integer and the coefficients satisfy $A, A_0 \geq 0$, $C, \beta_\mu, \delta_\nu > 0$ for all μ, ν . The converse is also true: every function of this form generates (i.e. its Laurent coefficients give) a doubly infinite totally positive sequence.

In the products and sums with inequalities in limits, we assume that the indexing variable changes in \mathbb{Z} or in some finite or infinite subinterval of \mathbb{Z} , and that it additionally satisfies the indicated inequalities. Accordingly, a product or sum can be empty, finite or infinite. Note that the indexation of four-way infinite matrices affects the multiplication. Here we adopt the following convention: the uppermost row and the leftmost column, which appear in representations of such matrices, have the index 1 unless another is stated explicitly.

The so-called Hurwitz-type matrices have applications to stability theory. They are built from two Toeplitz matrices; more specifically,

Definition. The *Hurwitz-type matrix* is a matrix of the form

$$H(p, q) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ \dots & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ \dots & a_{-1} & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ \dots & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ \dots & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2)$$

where $p(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ and $q(z) = \sum_{k=-\infty}^{\infty} b_k z^k$ are formal power series.

Recent publications [HT2012, Dy2014] have shown that a criterion relevant to Theorem 1 holds for the Hurwitz-type matrices. The main goal of the present study is to give an extension of that criterion: to determine conditions on the power series $p(z)$ and $q(z)$ necessary and sufficient for total nonnegativity of the matrix $H(p, q)$. Like in the earlier studied cases, one of the conditions is that the ratio $\frac{q(z)}{p(z)}$ maps the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ into itself. To give a more precise statement, let us introduce the following class of functions:

Definition. A function $F(z)$ is called an \mathcal{S} -function if it is holomorphic and satisfies $\operatorname{Im} z \cdot \operatorname{Im} F(z) \geq 0$ for all $z \not\equiv 0$ and if additionally $F(z) \geq 0$ wherever $z > 0$.

The straightforward corollary of the definition is that $F(\bar{z}) = \overline{F(z)}$ for each \mathcal{S} -function $F(z)$ wherever it is regular. We need a subclass of \mathcal{S} -functions introduced in the following lemma.

Lemma 2. Let $p(z)$ and $q(z)$ be two functions of the form (1); then their ratio $F(z) = \frac{q(z)}{p(z)}$ is an \mathcal{S} -function if and only if there exists a function $g(z)$ of the form (1), such that

$$\frac{p(z)}{g(z)} = a_0 \prod_{\nu>0} \left(1 + \frac{z}{\alpha_\nu}\right) \prod_{\nu<0} \left(1 + \frac{z^{-1}}{\alpha_\nu}\right), \quad \frac{q(z)}{g(z)} = b_0 \prod_{\mu>0} \left(1 + \frac{z}{\beta_\mu}\right) \prod_{\mu<0} \left(1 + \frac{z^{-1}}{\beta_\mu}\right) \quad \text{and} \\ 0 < \dots < \alpha_{-2}^{-1} < \beta_{-1}^{-1} < \alpha_{-1}^{-1} < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \dots;$$

if the sequence of μ terminates on the left at μ_0 , then β_{μ_0} can be positive or zero² and the sequence of ν also terminates on the left at μ_0 .

This lemma is an analogue of a theorem due to Kreĭn, see [Lev64, p. 308]. In other words, under its conditions the function $F(z)$ can be expressed as presented below in (13) or (14). The chain inequality means that zeros of $\frac{p(z)}{g(z)}$ and $\frac{q(z)}{g(z)}$ are *interlacing*, that is all zeros of each of the functions are real and separated by zeros of another. Lemma 2 provides another reformulation of the item (a) in our main result:

Theorem 3. *If $a_0 \neq 0$, then the following conditions are equivalent:*

- (a) *The series $p(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ and $q(z) = \sum_{k=-\infty}^{\infty} b_k z^k$ converge in some common annulus to functions of the form (1) and their ratio $F(z) = \frac{q(z)}{p(z)}$ is an \mathcal{S} -function.*
- (b) *The matrix $H(p, q)$ is totally nonnegative and there exists $k \in \mathbb{Z}$ such that $a_k^2 \neq a_{k-1} a_{k+1}$.*

Remark 4. The conditions $a_0 \neq 0$ and $a_k^2 = a_{k-1} a_{k+1}$ for all k are equivalent to the condition $a_k = a_0^{1-k} a_1^k \neq 0$ for all k . This case is excluded in (b) of Theorem 3 as corresponding to the divergence of the power series $p(z)$ and, unless $q(z) \equiv 0$, of the series $q(z)$ by Lemma 14 (in fact, we have $a_0 q(z) \equiv b_0 p(z)$).

Both papers [HT2012, Dy2014] exploit a relation to the matching moment problem through the Hurwitz transform (see e.g. [ChM49, p. 44] or [HT2012, p. 427]). In turn, doubly infinite series do not allow conducting the same procedure due to the lack of the matching moment problem. To get around the difficulty, we apply a modification of the technique [AESW51, Kar68] developed in Section 3 for factoring out a totally nonnegative Toeplitz matrix from a totally nonnegative Hurwitz-type matrix. Another key point is Lemma 20, which characterizes the function corresponding to the resulting Hurwitz-type matrix by extending a fact known for polynomials, see e.g. [Wa2000, Lemma 3.4]. The latter step is based on simple but effective Lemma 19; the converse to this lemma is proved in [Dy2016].

By writing that a function has one of the above representations, we assume that the involved products are locally uniformly convergent unless the converse is stated explicitly. In the above theorems, the convergence follows from the total nonnegativity of the involved matrices. The condition of convergence is well-known and can be expressed as the following theorem.

Theorem 5 (see e.g. [Lev64, pp. 7–13, 21]). *The infinite product $\prod_{v=0}^{\infty} \left(1 + \frac{z}{\alpha_v}\right)$, converges uniformly in z varying in compact subsets of \mathbb{C} if and only if the series $\sum_{v=0}^{\infty} \frac{1}{|\alpha_v|}$ converges. If so, then for any $\varepsilon > 0$ the estimates $\prod_{v=0}^{\infty} \left|1 + \frac{z}{\alpha_v}\right| < C e^{\varepsilon R}$ and, outside exceptional disks with an arbitrarily small sum of radii, $\prod_{v=0}^{\infty} \left|1 + \frac{z}{\alpha_v}\right| > C e^{-\varepsilon R}$ provided that $|z| \leq R$ and the positive numbers R and C are big enough.*

2 Basic properties of infinite Toeplitz and Hurwitz-type matrices

An important property of totally positive sequences is that they have no gaps, i.e. no zero coefficients between non-zero coefficients:

Lemma 6. *If $p(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ and the matrix $T(p)$ is totally nonnegative, then $a_k = 0 \neq a_{k+1} \implies a_{k-n} = 0$ and $a_k = 0 \neq a_{k-1} \implies a_{k+n} = 0$ for all $n \in \mathbb{Z}_{>0}$.*

In other words, if any of the coefficients of $p(z)$ turns to zero, then all coefficients to the left or to the right must be zero. In this case, $p(z)$ is either a Laurent polynomial or a singly infinite series with no gaps. Lemma 7 exploits an analogous property of minors of $T(p)$.

²When $\beta_{\mu_0} = 0$, the corresponding factor $\left(1 + \frac{z}{\beta_{\mu_0}}\right)$ needs to be replaced by the factor z .

Proof of Lemma 6. Note that all coefficients of $p(z)$ are nonnegative. If for some $k \in \mathbb{Z}$ the coefficient a_k is zero and the neighbouring one a_{k-1} is nonzero, then for each integer $n \geq k$ the inequality

$$0 \leq \begin{vmatrix} a_k & a_{n+1} \\ a_{k-1} & a_n \end{vmatrix} = -a_{n+1}a_{k-1} \leq 0$$

yields that $a_{n+1} = 0$. Conversely, if $a_n = 0 \neq a_{n+1}$ for some n , then from the same inequality we have $a_{k-1} = 0$ for all $k \leq n$. \square

Lemma 7. *If a series $p(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ is such that the matrix $T(p)$ is totally nonnegative, and $0 \neq a_{k-1}^2 = a_k a_{k-2}$ for some integer k , then $p(z)$ does not converge to any function holomorphic in $\mathbb{C} \setminus \{0\}$.*

Proof. The condition $0 \neq a_{k-1}^2 = a_k a_{k-2}$ implies that a_k, a_{k-1} and a_{k-2} are nonzero. Unless $a_j = a_0^{1-j} a_1^j$ for all j , there exists an integer i such that $a_{i-1}a_k \neq a_{k-1}a_i$ (see Remark 4). Therefore, assuming $i > j := k$ gives us the following relation:

$$\begin{aligned} 0 \leq \begin{vmatrix} a_{i-2} & a_{i-1} & a_i \\ a_{j-2} & a_{j-1} & a_j \\ a_{j-3} & a_{j-2} & a_{j-1} \end{vmatrix} &= \frac{1}{a_{j-1}} \begin{vmatrix} a_{i-2} & a_{i-1} & a_i \\ a_{j-2} & a_{j-1} & a_j \\ a_{j-1}a_{j-3} - a_{j-2}^2 & 0 & 0 \end{vmatrix} \\ &= -a_{j-1}^{-1} (a_{i-1}a_j - a_i a_{j-1}) (a_{j-2}^2 - a_{j-1}a_{j-3}) \leq 0. \end{aligned} \quad (3)$$

Consequently, $a_{j-2}^2 = a_{j-1}a_{j-3} \neq 0$. Furthermore, $a_{i-1}a_{j-1} - a_i a_{j-2} = \frac{a_{j-1}}{a_j} (a_{i-1}a_j - a_i a_{j-1}) \neq 0$, and hence (3) is valid for $j = k-1$. Sequentially letting $j = k-1, k-2, \dots$ in (3) thus implies that $a_{j-1}^2 = a_j a_{j-2} \neq 0$ for each $j < k$. According to the ratio test, the series $p(z)$ diverges in the disk of the radius $\lim_{j \rightarrow -\infty} \frac{a_{j-1}}{a_j} = \frac{a_{k-1}}{a_k} > 0$. The proof for the case $i+1 < j := k$ is analogous: since

$$0 \leq \begin{vmatrix} a_{j-1} & a_j & a_{j+1} \\ a_{j-2} & a_{j-1} & a_j \\ a_{i-1} & a_i & a_{i+1} \end{vmatrix} = \frac{1}{a_j} \begin{vmatrix} a_{j-1} & a_j & a_{j+1} \\ 0 & 0 & a_j^2 - a_{j+1}a_{j-1} \\ a_{i-1} & a_i & a_{i+1} \end{vmatrix} = -a_j^{-1} (a_i a_{j-1} - a_{i-1}a_j) (a_j^2 - a_{j+1}a_{j-1}) \leq 0,$$

the equality $a_j^2 = a_{j+1}a_{j-1} \neq 0$ holds. Due to $a_{i-1}a_{j+1} - a_i a_j = \frac{a_{j+1}}{a_j} (a_{i-1}a_j - a_i a_{j-1}) \neq 0$, the same result is true for each $j = k+1, k+2, \dots$. Consequently, the series $p(z)$ has the radius of convergence equal to $\lim_{j \rightarrow +\infty} \frac{a_{j-1}}{a_j} = \frac{a_{k-1}}{a_k} > 0$. \square

In fact, applying a version of the Sylvester determinant identity (see e.g. [Pi2010, formula (5.1), p. 136]) reduces the proof of Lemma 7 to the proof of Lemma 6. The following lemma is an analogue of Lemma 6 for Hurwitz-type matrices.

Lemma 8. *Let $H(p, q)$ be totally nonnegative. If both series are non-trivial and one of them has a zero coefficient, then both series terminate on the same side. More specifically,*

- $a_k = 0 \neq a_{k-1} \implies b_{k+r} = 0$,
- $a_k = 0 \neq a_{k+1} \implies b_{k-r+1} = 0$,
- $b_k = 0 \neq b_{k-1} \implies a_{k+r-1} = 0$,
- $b_k = 0 \neq b_{k+1} \implies a_{k-r} = 0$

for all $r = 1, 2, \dots$. In addition, $a_{k-1}, a_k \neq 0 \implies b_k \neq 0$ and $b_k, b_{k+1} \neq 0 \implies a_k \neq 0$.

Proof. By Lemma 6, the condition $a_k = 0 \neq a_{k-1}$ for some k yields $a_{k+r-1} = 0$; therefore,

$$0 \leq \begin{vmatrix} b_k & b_{k+r} \\ a_{k-1} & a_{k+r-1} \end{vmatrix} = -a_{k-1}b_{k+r} \leq 0 \implies b_{k+r} = 0.$$

The next three implications follow analogously from evaluating (respectively) the minors

$$\begin{vmatrix} a_{k-r+1} & a_{k+1} \\ b_{k-r+1} & b_{k+1} \end{vmatrix}, \quad \begin{vmatrix} a_{k-1} & a_{k+r-1} \\ b_{k-1} & b_{k+r-1} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_{k-r+1} & b_{k+1} \\ a_{k-r} & a_k \end{vmatrix}.$$

The assumption $a_{k-1}, a_k \neq 0 = b_k$ gives that the minors

$$\begin{vmatrix} a_{k-r} & a_k \\ b_{k-r} & b_k \end{vmatrix} = -a_k b_{k-r} \quad \text{and} \quad \begin{vmatrix} b_k & b_{k+r} \\ a_{k-1} & a_{k+r-1} \end{vmatrix} = -a_{k-1} b_{k+r}$$

vanish for all $r = 1, 2, \dots$, which is inconsistent with the non-triviality of $q(z)$. Similarly, $b_k, b_{k+1} \neq 0 = a_k$ gives vanishing of

$$\begin{vmatrix} b_{k-r+1} & b_{k+1} \\ a_{k-r} & a_k \end{vmatrix} = -a_{k-r} b_{k+1} \quad \text{and} \quad \begin{vmatrix} a_k & a_{k+r} \\ b_k & b_{k+r} \end{vmatrix} = -b_k a_{k+r}$$

for all $r = 1, 2, \dots$, which contradicts to the non-triviality of $p(z)$. \square

The next fact is relevant to Lemma 7 for series terminating on the right, and its proof can also be conducted with the help of the Sylvester determinant identity. Note that Lemma 9 admits a reformulation for series terminating on the left.

Lemma 9. Given series $p(z) = \sum_{k=-\infty}^n a_k z^k$ such that $a_n \neq 0$ and $q(z) = \sum_{k=-\infty}^{\infty} b_k z^k \neq 0$ suppose that $H(p, q)$ is totally nonnegative and $a_{k-1}b_k = a_k b_{k-1} \neq 0$ for some $k < n$. Then

- (a) the inequality $a_{n-1}b_n \neq a_n b_{n-1}$ implies that neither $p(z)$ nor $q(z)$ can converge to any function holomorphic in $\mathbb{C} \setminus \{0\}$;
- (b) if at least one of the series $p(z), q(z)$ converges to a function holomorphic in $\mathbb{C} \setminus \{0\}$, then $p(z) = \frac{a_n}{b_n} q(z)$.

Proof. By Lemma 6, if $a_{k-1}, a_n \neq 0$, then all the numbers a_{k-1}, a_k, \dots, a_n are strictly positive. Since the last term $-a_j b_{k-1}$ in the right-hand side of

$$0 \leq \begin{vmatrix} a_{k-1} & a_j \\ b_{k-1} & b_j \end{vmatrix} = a_{k-1}b_j - a_j b_{k-1},$$

is nonzero for each $j = k, k+1, \dots, n$, all the numbers b_{k-1}, b_k, \dots, b_n are strictly positive as well. Furthermore, Lemma 8 yields $b_{n+2} = b_{n+3} = \dots = 0$. The condition $a_{k-1}b_k = a_k b_{k-1} \neq 0$ implies that

$$0 \leq \begin{vmatrix} a_{k-1} & a_k & a_n \\ b_{k-1} & b_k & b_n \\ a_{k-2} & a_{k-1} & a_{n-1} \end{vmatrix} = \begin{vmatrix} a_{k-1} & a_k & a_n \\ 0 & 0 & b_n - \frac{b_{k-1}}{a_{k-1}} a_n \\ a_{k-2} & a_{k-1} & a_{n-1} \end{vmatrix} = \left(\frac{b_{k-1}}{a_{k-1}} a_n - b_n \right) \begin{vmatrix} a_{k-1} & a_k \\ a_{k-2} & a_{k-1} \end{vmatrix} \leq 0 \quad (4)$$

and, due to $a_{n+1} = 0$, for each $i \leq k-2$

$$0 \leq \begin{vmatrix} a_{i+n+1-k} & a_n & a_{n+1} \\ a_i & a_{k-1} & a_k \\ b_i & b_{k-1} & b_k \end{vmatrix} = -a_n(a_i b_k - a_k b_i) \leq 0 \implies a_i b_k = a_k b_i. \quad (5)$$

The last equality shows that $a_i \neq 0 \iff b_i \neq 0$ for all $i \leq k-2$ since a_k and b_k are nonzero. Moreover, (5) implies that the series $p(z)$ and $q(z)$ converge in the same domain:

$$q(z) = \frac{b_k}{a_k} p(z) + \sum_{i=k}^n \left(b_i - \frac{b_k}{a_k} a_i \right) z^i + b_{n+1} z^{n+1}.$$

Let us prove (a). Multiplying $\frac{a_{k-1}}{a_{n-1}} \geq \frac{b_{k-1}}{b_{n-1}}$ and $a_{n-1} b_n > b_{n-1} a_n$ gives the inequality $a_{k-1} b_n > a_n b_{k-1}$. Substituting it into the relation (4) yields $a_{k-1}^2 = a_k a_{k-2}$, so the series $p(z)$ does not converge to any function holomorphic in $\mathbb{C} \setminus \{0\}$ by Lemma 7. Neither does the series $q(z)$, because it coincides with $\frac{b_k}{a_k} p(z)$ up to a Laurent polynomial.

To prove (b), note that the convergence of $p(z)$ for $z \neq 0$ implies $a_{k-1}^2 \neq a_k a_{k-2}$ by Lemma 7. Due to (4), the coefficients of $p(z)$ and $q(z)$ satisfy $a_{k-1} b_n = a_n b_{k-1}$. We will get a contradiction to this equality whenever $a_i b_n > a_n b_i$ for some $i = k, k+1, \dots, n-1$:

$$a_{k-1} b_n = \frac{a_{k-1}}{a_i} a_i b_n > \frac{a_{k-1}}{a_i} a_n b_i \geq a_n b_{k-1}$$

since $a_{k-1} b_i \geq b_{k-1} a_i$. Summing up, $a_i b_n = a_n b_i$ for each $i < n$. The coefficient b_{n+1} is zero, because

$$0 \leq \begin{vmatrix} a_{n-1} & a_n & a_{n+1} \\ b_{n-1} & b_n & b_{n+1} \\ a_{n-2} & a_{n-1} & a_n \end{vmatrix} = -b_{n+1} \begin{vmatrix} a_{n-1} & a_n \\ a_{n-2} & a_{n-1} \end{vmatrix} \leq 0.$$

Consequently, $q(z) = \frac{b_n}{a_n} p(z)$. □

Theorem 10 (Whitney, see [Whi52]). *Let m, n, j be positive integers such that $j \leq m$ and let $\{a_{ik}\}_{1 \leq i \leq m, 1 \leq k \leq n}$ be real numbers. The following matrices are totally nonnegative simultaneously:*

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 1 & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{j2} & a_{j3} & \dots & a_{jn} \\ 0 & a_{j+1,2} & a_{j+1,3} & \dots & a_{j+1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{12} & a_{13} & \dots & a_{1n} \\ a_{22} - a_{12} & a_{23} - a_{13} & \dots & a_{2n} - a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j2} - a_{j-1,2} & a_{j3} - a_{j-1,3} & \dots & a_{jn} - a_{j-1,n} \\ a_{j+1,2} & a_{j+1,3} & \dots & a_{j+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}.$$

Lemma 11 (e.g., [Pi2010, p. 6]). *Given an $n \times m$ matrix M let \widetilde{M} be the matrix obtained from M by permuting its rows and columns in the opposite order. Then M and \widetilde{M} are totally nonnegative simultaneously.*

Proof. Given an arbitrary positive integer $k \leq \min\{n, m\}$ and any two sets of integers $0 < i_1 < i_2 < \dots < i_k \leq n$ and $0 < j_1 < \dots < j_k \leq m$ denote the minor of the matrix M with rows i_1, i_2, \dots, i_k and columns j_1, j_2, \dots, j_k by

$$M \begin{pmatrix} i_1, i_2, \dots, i_k \\ j_1, j_2, \dots, j_k \end{pmatrix}.$$

Then the lemma follows from the identity

$$M \begin{pmatrix} i_1, i_2, \dots, i_k \\ j_1, j_2, \dots, j_k \end{pmatrix} = \left((-1)^{\frac{k(k-1)}{2}} \right)^2 \cdot \widetilde{M} \begin{pmatrix} n - i_k, n - i_{k-1}, \dots, n - i_1 \\ m - j_k, m - j_{k-1}, \dots, m - j_1 \end{pmatrix}.$$

□

Corollary 12. Let $p(z)$ and $q(z)$ be two power series, and let $\check{p}(z) := p(\frac{1}{z})$ and $\check{q}(z) := q(\frac{1}{z})$. Then the matrices $H(p, q)$ and $H(\check{q}, \check{p})$ are totally nonnegative simultaneously.

Proof. Indeed, assume that $p(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $q(z) = \sum_{n=-\infty}^{\infty} b_n z^n$. Then $\check{p}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$ and $\check{q}(z) = \sum_{n=-\infty}^{\infty} b_{-n} z^n$, so each submatrix of $H(\check{q}, \check{p})$ coincides with some submatrix of $H(p, q)$ up to the permutation of rows and columns in the opposite order. □

Lemma 13. Let $p(z)$ and $q(z)$ be two power series, and let $\tilde{p}(z) = zp(z)$. Then the matrices $H(p, q)$ and $H(q, \tilde{p})$ coincide up to a shift in indexation.

Proof. Denote $p(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ and $q(z) = \sum_{k=-\infty}^{\infty} b_k z^k$. On the one hand, $\tilde{p}(z) = \sum_{n=-\infty}^{\infty} a_{n-1} z^n$ and, hence,

$$H(q, \tilde{p}) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ \dots & a_{-1} & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ \dots & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ \dots & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & \dots \\ \dots & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & \dots \\ \dots & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

On the other hand, shifting the whole matrix $H(p, q)$ up results in the same matrix; that is, $H(q, \tilde{p})$ can be obtained by increasing³ the indices of rows in $H(p, q)$ by 1. □

3 Poles and exponential factors

If one of the series $p(z)$ or $q(z)$ is trivial, it converges in the whole plane; in this special case the total nonnegativity of $H(p, q)$ does not imply that another series converges in the same domain. For non-trivial series, the following lemma shows that the total nonnegativity of $H(p, q)$ yields the same annulus of convergence for both $p(z)$ and $q(z)$.

Lemma 14. Let power series $p(z)$ and $q(z)$ be non-trivial. If the matrix $H(p, q)$ is totally nonnegative, then the series converge in the same annulus, say $\{z \in \mathbb{C} : 0 \leq r < |z| < R \leq \infty\}$. If $a_k \neq 0$ for all $k \ll 0$, then

$$\lim_{k \rightarrow -\infty} \frac{b_k}{b_{k+1}} = \lim_{k \rightarrow -\infty} \frac{a_k}{a_{k+1}} = r;$$

if $a_k \neq 0$ for all $k \gg 0$, then

$$\lim_{k \rightarrow \infty} \frac{b_k}{b_{k+1}} = \lim_{k \rightarrow \infty} \frac{a_k}{a_{k+1}} = R.$$

The annulus can be empty, then for all k the coefficients satisfy $a_k = a_0^{1-k} a_1^k$ and $b_k = \frac{b_0}{a_0} a_k = b_0^{1-k} b_1^k$, i.e. all minors of $H(p, q)$ of order ≥ 2 vanish.

Proof. By Lemmata 6 and 8, there are two mutually exclusive possibilities: $\exists n \in \mathbb{Z}$ such that $a_k = b_k = 0$ for all $k > n$, or $\exists n \in \mathbb{Z}$ such that $a_k, b_k > 0$ for all $k > n$. In the former case, both series $p(z)$ and $q(z)$ converge

³There are infinitely many equivalent ways to describe this shift in indexation, because the matrix $H(p, q)$ does not alter when the indices of columns change by k and, simultaneously, the indices of rows change by $2k$ for any integer k .

outside some disk $|z| \leq r < \infty$ and we put $R = \infty$. In the latter case,

$$\begin{vmatrix} a_k & a_{k+1} \\ b_k & b_{k+1} \end{vmatrix} = a_k b_{k+1} - b_k a_{k+1} \geq 0 \quad \text{and} \quad \begin{vmatrix} b_k & b_{k+1} \\ a_{k-1} & a_k \end{vmatrix} = a_k b_k - a_{k-1} b_{k+1} \geq 0$$

together give

$$\frac{a_{k-1}}{a_k} \leq \frac{b_k}{b_{k+1}} \leq \frac{a_k}{a_{k+1}} \quad (6)$$

for all $k > n$, i.e. the limits

$$\lim_{k \rightarrow \infty} \frac{a_k}{a_{k+1}} = \lim_{k \rightarrow \infty} \frac{b_k}{b_{k+1}} =: R \in \left[\frac{a_{n+1}}{a_{n+2}}, +\infty \right) \subset (0, +\infty] \quad (7)$$

exist. By the ratio test, the radius of convergence of the series $\sum_{k=1}^{\infty} a_k z^k$ and $\sum_{k=1}^{\infty} b_k z^k$ is equal to R .

Analogously, there are two mutually exclusive possibilities: $\exists n \in \mathbb{Z}$ such that $a_k = b_k = 0$ for all $k \leq n$, or $\exists n \in \mathbb{Z}$ such that $a_k, b_k > 0$ for all $k \leq n$. In the former case, both series $p(z)$ and $q(z)$ converge in the disk $|z| < R$, that is $r = 0$. In the latter case, the inequality (6) holds provided that $k < n$, that is

$$\lim_{k \rightarrow -\infty} \frac{a_k}{a_{k+1}} = \lim_{k \rightarrow -\infty} \frac{b_k}{b_{k+1}} =: r \in \left[0, \frac{a_{n-1}}{a_n} \right) \subset [0, +\infty). \quad (8)$$

So, the ratio test implies that $\sum_{k=-\infty}^0 a_k z^k$ and $\sum_{k=-\infty}^0 b_k z^k$ converge absolutely outside the disk $|z| \leq r$. As a result, $p(z)$ and $q(z)$ converge absolutely if and only if $r < |z|$ and $|z| < R$ simultaneously.

Now, note that any of the equalities $r = 0$ and $R = \infty$ yields $r < R$. If $r > 0$ and $R < \infty$, then all coefficients of the series $p(z)$ and $q(z)$ are positive by Lemma 6, and hence the inequality (6) holds for each integer k . Accordingly,

$$r = \lim_{k \rightarrow -\infty} \frac{a_k}{a_{k+1}} \leq \lim_{k \rightarrow \infty} \frac{a_k}{a_{k+1}} = R.$$

If $r = R$, then for all $k \in \mathbb{Z}$ we necessarily have

$$\frac{a_k}{a_{k+1}} = \frac{a_0}{a_1} = \frac{b_k}{b_{k+1}} = \frac{b_0}{b_1},$$

that is

$$a_k = a_{k-1} \frac{a_1}{a_0} = a_0 \left(\frac{a_1}{a_0} \right)^k = a_0^{1-k} a_1^k \quad \text{and} \quad b_k = b_{k-1} \frac{b_1}{b_0} = b_0 \left(\frac{a_1}{a_0} \right)^k = \frac{b_0}{a_0} a_k.$$

□

Lemma 15. *Let the matrix $H(p, q)$ be totally nonnegative, and let $r < |z| < R$ be the annulus of convergence of the series $p(z)$ and $q(z)$ provided by Lemma 14. Then the inequality $R < \infty$ implies that the matrix $H(p_1, q_1)$ with $p_1(z) := (1 - \frac{z}{R})p(z)$ and $q_1(z) := (1 - \frac{z}{R})q(z)$ is totally nonnegative, and the inequality $r > 0$ implies that the matrix $H(p_2, q_2)$ is totally nonnegative, where $p_2(z) := (1 - \frac{r}{z})p(z)$ and $q_2(z) := (1 - \frac{r}{z})q(z)$.*

Proof. Suppose that $R < \infty$ and consider an arbitrary submatrix of $H(p, q)$ of the following form:

$$\begin{pmatrix} a_m & a_{m+1} & \dots & a_{m+2n-1} & a_{m+2n} \\ a_0 & a_1 & \dots & a_{2n-1} & a_{2n} \\ b_0 & b_1 & \dots & b_{2n-1} & b_{2n} \\ a_{-1} & a_0 & \dots & a_{2n-2} & a_{2n-1} \\ b_{-1} & b_0 & \dots & b_{2n-2} & b_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{-n+1} & b_{-n+2} & \dots & b_n & b_{n+1} \end{pmatrix},$$

where $m, n \in \mathbb{Z}_{>0}$. This matrix is totally nonnegative, and $a_m \neq 0$, $b_m \neq 0$ when m is big enough (see Lemma 8). Divide the first row by a_m and let $m \rightarrow +\infty$. Then Lemma 14 gives the relation

$$\frac{a_{m+n}}{a_m} = \frac{a_{m+1}}{a_m} \cdot \frac{a_{m+2}}{a_{m+1}} \cdots \frac{a_{m+n}}{a_{m+n-1}} \rightarrow R^{-n}$$

implying that

$$\begin{pmatrix} 1 & \frac{a_{m+1}}{a_m} & \cdots & \frac{a_{m+2n-1}}{a_m} & \frac{a_{m+2n}}{a_m} \\ a_0 & a_1 & \cdots & a_{2n-1} & a_{2n} \\ b_0 & b_1 & \cdots & b_{2n-1} & b_{2n} \\ a_{-1} & a_0 & \cdots & a_{2n-2} & a_{2n-1} \\ b_{-1} & b_0 & \cdots & b_{2n-2} & b_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{-n+1} & b_{-n+2} & \cdots & b_n & b_{n+1} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & R^{-1} & \cdots & R^{-2n+1} & R^{-2n} \\ a_0 & a_1 & \cdots & a_{2n-1} & a_{2n} \\ b_0 & b_1 & \cdots & b_{2n-1} & b_{2n} \\ a_{-1} & a_0 & \cdots & a_{2n-2} & a_{2n-1} \\ b_{-1} & b_0 & \cdots & b_{2n-2} & b_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{-n+1} & b_{-n+2} & \cdots & b_n & b_{n+1} \end{pmatrix} =: M,$$

where the convergence is entry-wise. The matrix M is totally nonnegative as an entry-wise limit of totally nonnegative matrices. Subtracting columns in M then gives

$$M_1 := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_0 & a_1 - a_0 R^{-1} & \cdots & a_{2n-1} - a_{2n-2} R^{-1} & a_{2n} - a_{2n-1} R^{-1} \\ b_0 & b_1 - b_0 R^{-1} & \cdots & b_{2n-1} - b_{2n-2} R^{-1} & b_{2n} - b_{2n-1} R^{-1} \\ a_{-1} & a_0 - a_{-1} R^{-1} & \cdots & a_{2n-2} - a_{2n-3} R^{-1} & a_{2n-1} - a_{2n-2} R^{-1} \\ b_{-1} & b_0 - b_{-1} R^{-1} & \cdots & b_{2n-2} - b_{2n-3} R^{-1} & b_{2n-1} - b_{2n-2} R^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{-n+1} & b_{-n+2} - b_{-n+1} R^{-1} & \cdots & b_{n-2} - b_{n-3} R^{-1} & b_{n-1} - b_{n-2} R^{-1} \end{pmatrix}.$$

If the matrix M is totally nonnegative, then by Theorem 10 the matrix M_1 is also totally nonnegative⁴. Now note that

$$p_1(z) = \left(1 - \frac{z}{R}\right) \sum_{m=-\infty}^{\infty} a_m z^m = \sum_{m=-\infty}^{\infty} a_{m+1} z^{m+1} - \sum_{m=-\infty}^{\infty} \frac{a_m}{R} z^{m+1} = \sum_{m=-\infty}^{\infty} (a_{m+1} - a_m R^{-1}) z^{m+1}$$

and $q_1(z) = \left(1 - \frac{z}{R}\right) \sum_{m=-\infty}^{\infty} b_m z^m = \sum_{m=-\infty}^{\infty} (b_{m+1} - b_m R^{-1}) z^{m+1},$

so any submatrix of $H(p_1, q_1)$ has only nonnegative minors, and hence $H(p_1, q_1)$ is itself totally nonnegative.

Analogously, suppose that $r > 0$. Then Lemmata 8 and 14 imply $\frac{b_m}{b_{m+n}} \rightarrow r^n$ as $m \rightarrow -\infty$, and therefore

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{2n-1} & a_{2n} \\ b_0 & b_1 & \cdots & b_{2n-1} & b_{2n} \\ a_{-1} & a_0 & \cdots & a_{2n-2} & a_{2n-1} \\ b_{-1} & b_0 & \cdots & b_{2n-2} & b_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{-n+1} & b_{-n+2} & \cdots & b_n & b_{n+1} \\ \frac{b_m}{b_{m+2n}} & \frac{b_{m+1}}{b_{m+2n}} & \cdots & \frac{b_{m+2n-1}}{b_{m+2n}} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} a_0 & a_1 & \cdots & a_{2n-1} & a_{2n} \\ b_0 & b_1 & \cdots & b_{2n-1} & b_{2n} \\ a_{-1} & a_0 & \cdots & a_{2n-2} & a_{2n-1} \\ b_{-1} & b_0 & \cdots & b_{2n-2} & b_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{-n+1} & b_{-n+2} & \cdots & b_n & b_{n+1} \\ r^{2n} & r^{2n-1} & \cdots & r & 1 \end{pmatrix},$$

⁴We apply Theorem 10 to the transpose of the matrices M and M_1 ; then we use the fact that the transposition does not affect the total nonnegativity.

where the convergence is entry-wise. Subtracting columns gives

$$M_2 := \begin{pmatrix} a_0 - ra_1 & a_1 - ra_2 & \dots & a_{2n-1} - ra_{2n} & a_{2n} \\ b_0 - rb_1 & b_1 - rb_2 & \dots & b_{2n-1} - rb_{2n} & b_{2n} \\ a_{-1} - ra_0 & a_0 - ra_1 & \dots & a_{2n-2} - ra_{2n-1} & a_{2n-1} \\ b_{-1} - rb_0 & b_0 - rb_1 & \dots & b_{2n-2} - rb_{2n-1} & b_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{-n+1} - rb_{-n+2} & b_{-n+2} - rb_{-n+3} & \dots & b_n - rb_{n+1} & b_{n+1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since the matrix $H(p, q)$ is totally nonnegative, the matrix M_2 is also totally nonnegative by Theorem 10 and Lemma 11. Crossing out the last column and row from M_2 gives a submatrix of $H(p_2, q_2)$, because

$$p_2(z) := \left(1 - \frac{r}{z}\right)p(z) = \sum_{k=-\infty}^{+\infty} (a_k - ra_{k+1})z^k \quad \text{and} \quad q_2(z) := \left(1 - \frac{r}{z}\right)q(z) = \sum_{k=-\infty}^{+\infty} (b_k - rb_{k+1})z^k.$$

The integer $n \geq 0$ is arbitrary, and thus the whole matrix of $H(p_2, q_2)$ is totally nonnegative. \square

Lemma 16. *If $H(p, q)$ is totally nonnegative and has a nonzero minor of order ≥ 2 , then the series $p(z)$ and $q(z)$ can be represented as $p(z) = p_*(z)g(z)$ and $q(z) = q_*(z)g(z)$, respectively. Here $g(z)$ denotes a function of the form (1) and the matrix $H(p_*, q_*)$ is totally nonnegative. Moreover, both $p_*(z)$ and $q_*(z)$ can be represented as the products*

$$\begin{aligned} p_*(z) &= C_1 z^j e^{Az + \frac{A_0}{z}} \prod_{v>0} \left(1 + \frac{z}{\alpha_v}\right) \prod_{v<0} \left(1 + \frac{z^{-1}}{\alpha_v}\right) \quad \text{and} \\ q_*(z) &= C_2 z^k e^{Az + \frac{A_0}{z}} \prod_{v>0} \left(1 + \frac{z}{\beta_v}\right) \prod_{v<0} \left(1 + \frac{z^{-1}}{\beta_v}\right), \end{aligned} \tag{9}$$

where $C_1, C_2, A, A_0 \geq 0$, the exponents $j, k \in \mathbb{Z}_{\geq 0}$ and $\alpha_v, \beta_v > 0$ for all $v \in \mathbb{Z}_{\neq 0}$.

Proof. If one of the series is trivial, then this theorem is equivalent to Theorem 1; therefore, we suppose below in this proof that both $p(z)$ and $q(z)$ are not trivial.

By Lemma 14, $p(z)$ and $q(z)$ converge in the same annulus $r < |z| < R$, that is $||z| - \rho| < (R - r)/2$, where $\rho := (R + r)/2$. The annulus is not empty, because there exists a nonzero minor of $H(p, q)$ of order ≥ 2 . Since the matrices $T(p)$ and $T(q)$ are totally nonnegative as submatrices of $H(p, q)$, both series $p(z)$ and $q(z)$ converge to functions of the form (1) by Theorem 1. In particular, the poles of these functions are positive and can condense only at $z = 0$ or infinity. Let us enumerate all their *common* poles as $\gamma_0, \gamma_1, \dots, \gamma_N$ excluding a possible pole at the origin, so that

$$1 < \max \left\{ \frac{\rho}{\gamma_i}, \frac{\gamma_i}{\rho} \right\} \leq \max \left\{ \frac{\rho}{\gamma_{i+1}}, \frac{\gamma_{i+1}}{\rho} \right\}$$

for all $i < N \leq \infty$ and each pole occurs only once. Denote

$$p_n(z) := p(z) \prod_{i=0}^n E_i^{m_i}(z) \quad \text{and} \quad q_n(z) := q(z) \prod_{i=0}^n E_i^{m_i}(z), \quad \text{where} \quad E_i(z) := \begin{cases} 1 - \frac{z}{\gamma_i} & \text{if } \gamma_i > \rho, \\ 1 - \frac{\gamma_i}{z} & \text{if } \gamma_i < \rho \end{cases}$$

and m_i stands for the order⁵ of the pole γ_i .

⁵In other words, m_i is the minimal number, such that at least one of the functions $p(z)E_i^{m_i}(z)$ and $q(z)E_i^{m_i}(z)$ is regular at the point $z = \gamma_i$.

Let $n = 0$. On the one hand, both $p_0(z)$ and $q_0(z)$ represent functions of the form (1), at least one of which has no pole at $\gamma_0 \in \{r, R\}$. On the other hand, these series converge in the same annulus by Lemma 15 since the matrix $H(p_0, q_0)$ is totally nonnegative. Consequently, neither of $p_0(z)$ and $q_0(z)$ has a pole at γ_0 , and $r < |z| < R$ is strictly nested in the annulus of convergence of these series. By induction on $n = 0, 1, \dots, N$ we obtain that the matrix $H(p_n, q_n)$ is totally nonnegative for each n , and hence the orders of the pole γ_n of $p(z)$ and of $q(z)$ coincide.

Both $p(z)$ and $q(z)$ can be represented as in (1), so the product $\prod_{i=0}^N E_i^{m_i}(z)$ converges in $\mathbb{C} \setminus \{0\}$ locally uniformly when $N = \infty$. Therefore, $p_n(z) \rightarrow p_*(z) := p(z) \prod_{i=0}^N E_i^{m_i}(z)$ and $q_n(z) \rightarrow q_*(z) := q(z) \prod_{i=0}^N E_i^{m_i}(z)$, and the Laurent coefficients converge as well. Moreover, both functions $p_*(z)$ and $q_*(z)$ are holomorphic in $\mathbb{C} \setminus \{0\}$: they have the form

$$\begin{aligned} p_*(z) &= C_1 z^j e^{Az + \frac{A_0}{z}} \cdot \prod_{v>0} \left(1 + \frac{z}{\alpha_v}\right) \prod_{v<0} \left(1 + \frac{z^{-1}}{\alpha_v}\right) \quad \text{and} \\ q_*(z) &= C_2 z^k e^{Bz + \frac{B_0}{z}} \cdot \prod_{\mu>0} \left(1 + \frac{z}{\beta_\mu}\right) \prod_{\mu<0} \left(1 + \frac{z^{-1}}{\beta_\mu}\right), \end{aligned} \quad (10)$$

where $A, A_0, B, B_0, C_1, C_2 \geq 0$; $j, k \in \mathbb{Z}$ and $\alpha_v, \beta_\mu > 0$ for all v, μ . The corresponding Hurwitz-type matrix $H(p_*, q_*)$ is totally nonnegative as the entry-wise limit of the matrices $H(p_n, q_n)$ as $n \rightarrow N$. If $p_*(z)$ is a Laurent polynomial, then $q_*(z)$ is also a Laurent polynomial by Lemma 8 and both $p_*(z)$ and $q_*(z)$ have only negative zeros by Theorem 1; in this case the assertion of the lemma holds true with $g(z) = \prod_{i=0}^N E_i^{-m_i}(z)$, so below we suppose that $p_*(z)$ has infinitely many nonzero coefficients.

To keep the notation, assume that $p(z) = p_*(z)$ and $q(z) = q_*(z)$, so that both series converge in $\mathbb{C} \setminus \{0\}$. According to Lemma 8, we can fix an integer k such that $a_{k-1}, a_k, a_{k+1} \neq 0$, and thus $b_k, b_{k+1} \neq 0$. The conditions

$$0 < p(x) = \max_{|z|=x} |p(z)| < \infty \quad \text{and} \quad 0 < q(x) = \max_{|z|=x} |q(z)| < \infty$$

hold for $x > 0$ since both series $p(x)$ and $q(x)$ have only nonnegative coefficients. From the total nonnegativity of $H(p, q)$ we obtain the inequalities $a_{n-1} \frac{b_{k+1}}{a_k} \geq b_n$ and $b_n \geq a_n \frac{b_k}{a_k}$ for all $n > k$, which are giving

$$x \frac{b_{k+1}}{a_k} \sum_{n=k}^{\infty} a_n x^n \geq \sum_{n=k+1}^{\infty} b_n x^n \geq \frac{b_k}{a_k} \sum_{n=k+1}^{\infty} a_n x^n \quad \text{for } x > 0.$$

Analogously, the inequalities $a_n \frac{b_k}{a_k} \geq b_n$ and $b_n \geq a_{n-1} \frac{b_{k+1}}{a_k}$ for $n \leq k$ give

$$\frac{b_k}{a_k} \sum_{n=-\infty}^k a_n x^n \geq \sum_{n=-\infty}^k b_n x^n \geq x \frac{b_{k+1}}{a_k} \sum_{n=-\infty}^{k-1} a_n x^n \quad \text{for } x > 0.$$

Summing up then yields

$$\begin{aligned} x \frac{b_{k+1}}{a_k} p(x) &\geq q(x) + O(x^k) \geq \frac{b_k}{a_k} p(x) \quad \text{and} \quad \frac{p(x)}{x^k} \geq a_{k+1} x \rightarrow +\infty \quad \text{as } x \rightarrow +\infty, \\ x \frac{b_{k+1}}{a_k} p(x) &\leq q(x) + O(x^{k+1}) \leq \frac{b_k}{a_k} p(x) \quad \text{and} \quad \frac{x p(x)}{x^{k+1}} \geq \frac{a_{k-1}}{x} \rightarrow +\infty \quad \text{as } x \rightarrow +0. \end{aligned}$$

In other words, the big-O terms are neglectable and we can write

$$\lim_{x \rightarrow +\infty} \frac{q(x)}{x p(x)} = \lim_{x \rightarrow +\infty} \frac{C_2 x^k e^{Bz + \frac{B_0}{z}} \cdot \prod_{\mu>0} \left(1 + \frac{x}{\beta_\mu}\right) \prod_{\mu<0} \left(1 + \frac{x^{-1}}{\beta_\mu}\right)}{C_1 x^{j+1} e^{Az + \frac{A_0}{z}} \cdot \prod_{v>0} \left(1 + \frac{x}{\alpha_v}\right) \prod_{v<0} \left(1 + \frac{x^{-1}}{\alpha_v}\right)} \leq \frac{b_{k+1}}{a_k} \quad \text{and} \quad \lim_{x \rightarrow +\infty} \frac{q(x)}{p(x)} \geq \frac{b_k}{a_k} \quad (11)$$

with positive A, B, A_0, B_0 . Recall that $\frac{b_k}{a_k} > 0$, which implies $B = A$ since otherwise the dominant infinite products $\prod_{\mu>0} \left(1 + \frac{x}{\beta_\mu}\right)$ and $\prod_{\nu>0} \left(1 + \frac{x}{\alpha_\nu}\right)$ are neglectable with respect to the exponential term $e^{(B-A)x}$ as $x \rightarrow \infty$ by Theorem 5. Analogously,

$$\lim_{x \rightarrow +0} \frac{q(x)}{p(x)} = \lim_{x \rightarrow +0} \frac{C_2 x^k e^{\frac{B_0}{x}} \cdot \prod_{\mu>0} \left(1 + \frac{x}{\beta_\mu}\right) \prod_{\mu<0} \left(1 + \frac{x^{-1}}{\beta_\mu}\right)}{C_1 z^j e^{\frac{A_0}{z}} \cdot \prod_{\nu>0} \left(1 + \frac{x}{\alpha_\nu}\right) \prod_{\nu<0} \left(1 + \frac{x^{-1}}{\alpha_\nu}\right)} \leq \frac{b_k}{a_k} \quad \text{and} \quad \overline{\lim}_{x \rightarrow +0} \frac{q(x)}{xp(x)} \geq \frac{b_{k+1}}{a_k} \quad (12)$$

which implies $B_0 = A_0$ due to $\frac{b_{k+1}}{a_k} > 0$. Therefore, to finally prove the lemma it is enough to take $g(z) = \prod_{i=0}^N E_i^{-m_i}(z)$. \square

4 \mathcal{S} -functions

Lemma 17. *The product*

$$C \frac{\prod_{\mu>0} \left(1 + \frac{z}{\beta_\mu}\right) \prod_{\mu<0} \left(1 + \frac{z^{-1}}{\beta_\mu}\right)}{\prod_{\nu>0} \left(1 + \frac{z}{\alpha_\nu}\right) \prod_{\nu<0} \left(1 + \frac{z^{-1}}{\alpha_\nu}\right)}, \quad \text{where } C > 0, \text{ and the numbers} \quad (13)$$

$$0 < \dots < \alpha_{-2}^{-1} < \beta_{-1}^{-1} < \alpha_{-1}^{-1} < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \dots$$

satisfy $\sum_{\nu \neq 0} (\alpha_\nu^{-1} + \beta_\nu^{-1}) < \infty$, determines an \mathcal{S} -function. Analogously,

$$C \frac{z + \beta_0}{z + \alpha_0} \cdot \frac{\prod_{\mu>0} \left(1 + \frac{z}{\beta_\mu}\right)}{\prod_{\nu>0} \left(1 + \frac{z}{\alpha_\nu}\right)}, \quad (14)$$

where $C \geq 0$ and the numbers $0 \leq \beta_0 < \alpha_0 < \beta_1 < \alpha_1 < \dots$ satisfy $\sum_{\nu>0} (\alpha_\nu^{-1} + \beta_\nu^{-1}) < \infty$ is a meromorphic \mathcal{S} -function. Products over μ and ν in (13) or (14) can be terminating, in which case the numerator and the denominator retain to have interlacing zeros.

Proof. Suppose that $F(z)$ has the form (13) and denote

$$F_n(z) := C \frac{q_n(z)}{p_n(z)}, \quad \text{where} \quad q_n(z) = \prod_{\nu=1}^n \left(1 + \frac{z}{\beta_\nu}\right) \left(1 + \frac{z^{-1}}{\beta_{-\nu}}\right), \quad p_n(z) = \prod_{\nu=1}^n \left(1 + \frac{z}{\alpha_\nu}\right) \left(1 + \frac{z^{-1}}{\alpha_{-\nu}}\right). \quad (15)$$

Note that the product $\prod_{\nu=-n}^{-1} \frac{\alpha_\nu}{\beta_\nu} = \prod_{\nu=-n}^{-1} \frac{\beta_\nu^{-1}}{\alpha_\nu^{-1}} < 1$ is bounded. For each $n \in \mathbb{Z}_{>0}$, the rational function

$$F_n(z) = C \cdot \prod_{\nu=-n}^{-1} \frac{\alpha_\nu}{\beta_\nu} + \sum_{\nu=1}^n \left(\frac{A_{\nu,n} z}{z + \alpha_\nu} + \frac{A_{-\nu,n} z}{z + \frac{1}{\alpha_{-\nu}}} \right), \quad \text{where} \quad A_{\nu,n} = C \frac{q_n(z)}{z p'_n(z)} \Big|_{z = -\alpha_\nu^{\text{sign } \nu}} > 0, \quad (16)$$

is an \mathcal{S} -function as each of its partial fractions is such. The condition $\sum_{\nu \neq 0} (\alpha_\nu^{-1} + \beta_\nu^{-1}) < \infty$ implies the locally uniform convergence of each product in (13) (see Theorem 5) and, therefore, of the numerator $q_n(z)$ and the denominator $p_n(z)$ as $n \rightarrow \infty$. Since the denominator is nonzero for $z \neq 0$, the function $F(z)$ is the limit of $F_n(z)$ as $n \rightarrow \infty$ uniform on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. Moreover,

$$\text{Im } F(z) \cdot \text{Im } z = \lim_{n \rightarrow \infty} F_n(z) \cdot \text{Im } z \geq 0;$$

the inequality is strict outside the real line due to the maximum principle for the harmonic function $\text{Im } F(z)$.

The assertion that the expression (14) represents an \mathcal{S} -function follows by omitting from (16) terms that correspond to absent poles. \square

Lemma 18. *Given $F(z)$ of the form (13) or (14), the function $z^p F(z)$ with any integer $p \neq 0$ cannot be a mapping of \mathbb{C}_+ into itself provided that it is not equal identically to Cz or C ; the function $\frac{z}{F(z)}$ is an \mathcal{S} -function of the form (13) or (14).*

Proof. Let $-\alpha_v$ be an arbitrary pole of $F(z)$, then (16) implies that $F(z) \sim \frac{-A_v \alpha_v}{z + \alpha_v}$ for z close enough to $-\alpha_v$, where $A_v = \lim_{n \rightarrow \infty} A_{v,n} > 0$. If $z^p F(z)$ also maps the upper half of the complex plane into itself, then $(-\alpha_v)^{p+1} A_v < 0$, and hence $p \neq 0$ must be an even number. Nevertheless, if $z_* = \exp \frac{\pi}{|p|} i$, then $\text{Im } z_* > 0$ and $\text{Im } z_*^p F(z_*) = -\text{Im } F(z_*) < 0$.

For the second part of the lemma, it is enough to note that the reciprocal of the product (13) can be expressed as

$$\frac{1}{F(z)} = C \frac{\prod_{v>0} \left(1 + \frac{z}{\alpha_v}\right) \prod_{v<0} \left(1 + \frac{z^{-1}}{\alpha_v}\right)}{\prod_{\mu>0} \left(1 + \frac{z}{\beta_\mu}\right) \prod_{\mu<0} \left(1 + \frac{z^{-1}}{\beta_\mu}\right)} = \frac{C}{z\alpha_{-1}} \frac{(z\alpha_{-1} + 1) \prod_{v>0} \left(1 + \frac{z}{\alpha_v}\right) \prod_{v<-1} \left(1 + \frac{z^{-1}}{\alpha_v}\right)}{\prod_{\mu>0} \left(1 + \frac{z}{\beta_\mu}\right) \prod_{\mu<0} \left(1 + \frac{z^{-1}}{\beta_\mu}\right)}, \quad (17)$$

so the relabelling the $\beta_\mu \mapsto \tilde{\alpha}_\mu$ for all $\mu \neq 0$; $\alpha_{-1} \mapsto \tilde{\beta}_1^{-1}$ and $\alpha_v \mapsto \tilde{\beta}_{v-1}$ for all $v \notin \{0, 1\}$ yields that $\frac{z}{F(z)}$ has the form (13). An analogous reasoning works for the reciprocal of (14). \square

Proof of Lemma 2. On account of Lemma 17, it is enough to prove that the function $F(z)$ is an \mathcal{S} -function only if has the form (13) or (14). Since $F(z)$ is regular for $z > 0$, the poles of $p(z)$ and $q(z)$ coincide and have same orders. The Carathéodory inequality implies that the absolute value of a function $\phi(z)$ mapping the upper half of the complex plane into itself satisfies (see e.g. [ChM49, p. 71])

$$\frac{2}{c\rho} \leq |\phi(\rho e^{i\frac{\pi}{6}})| \leq 2c\rho, \quad (18)$$

where $c > 0$ is an appropriate constant and $\rho > 1$. All infinite products in (1) cannot grow or decrease at an exponential rate in $|z|$ (see Theorem 5). Thus, if $F(z)$ were able to have an exponential factor of the form $e^{\pm Az}$ with $A > 0$, then necessarily $|F(\rho e^{i\frac{\pi}{6}})| = |F(\rho \frac{\sqrt{3}}{2} + i \frac{\rho}{2})| \sim e^{\pm \frac{\sqrt{3}}{2} A \rho}$ as $\rho \gg 1$, which contradicts (18). The function $\frac{1}{F(1/z)}$ maps the upper half of the complex plane into itself; thus, it satisfies the inequality (18), which implies that an exponential factor of the form $e^{\pm A_0/z}$ in $F(z)$ with $A_0 > 0$ is absent. Summing up, the exponential factors in $p(z)$ and $q(z)$ must be the same.

Zeros and poles of $\frac{q(z)}{p(z)}$ interlace, because all its poles are simple and the residues are negative. Unless $F(z)$ is meromorphic, the order of zeros can be made as in (13) by taking out some power of z , cf. (17). The resulting power of the factor z in the representation of $F(z)$ then must be 0 by Lemma 18. The case of meromorphic $F(z)$ is more straightforward: no poles of $F(z)$ can appear between its maximal zero and the origin, otherwise $F(0+) < 0$ due to negativity of residues. \square

5 Total nonnegativity and interlacing zeros

Lemma 19. *If the matrix $H(p, q)$ is totally nonnegative and $\tilde{p}(z) := zp(z)$, then for arbitrarily taken nonnegative numbers A and B both matrices $T(Ap + Bq)$ and $T(Aq + B\tilde{p})$ are totally nonnegative.*

Proof. Observe that

$$T(Ap + Bq) = H^\top(A, B) H(p, q) \quad \text{and} \quad T(Aq + B\tilde{p}) = H^\top(A, B) H(q, \tilde{p}), \quad (19)$$

where the auxiliary totally nonnegative matrix $H^T(A, B)$ is the transpose of $H(A, B)$:

$$H^T(A, B) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & A & B & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & A & B & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & A & B & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (h_{ij})_{i,j=-\infty}^{\infty}, \quad \text{where} \quad h_{ij} = \begin{cases} A & \text{if } j = 2i - 1, \\ B & \text{if } j = 2i, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 13, the matrix $H(q, \tilde{p})$ is totally nonnegative whenever $H(p, q)$ is totally nonnegative. Therefore, applying the Cauchy-Binet formula to the expressions (19) yields that all minors of the matrices $T(Ap + Bq)$ and $T(Aq + B\tilde{p})$ must be nonnegative. \square

Lemma 20. *If $H(p, q)$ is totally nonnegative and has a nonzero minor of order ≥ 2 and, additionally, $p(z) \neq 0$, then the ratio $F(z) := \frac{q(z)}{p(z)}$ of functions represented by the series $p(z)$ and $q(z)$ has one of forms (13) or (14) or $C \cdot (z + \beta_0)$ with $C, \beta_0 \geq 0$.*

Proof. Lemma 16 establishes that $p(z) = p_*(z)g(z)$ and $q(z) = p_*(z)g(z)$, where the series $p_*(z)$ and $q_*(z)$ converge to functions of the form (9) and $g(z)$ can be represented as in (1). Furthermore, the matrix $H(p_*, q_*)$ is totally nonnegative. Since the common factor $g(z)$ does not affect the ratio $F(z)$, we assume that $p(z) = p_*(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ and $q(z) = q_*(z) = \sum_{k=-\infty}^{\infty} b_k z^k$ without loss of generality. In particular, both series $p(z)$ and $q(z)$ converge to functions for all $z \neq 0$ and we keep the same designations for the functions. Now note that the totally nonnegative Toeplitz matrices in Lemma 19 are constructed from the coefficients of the series $Ap(z) + Bq(z)$ and $Aq(z) + Bzp(z)$, and hence the limits of these series have the form (1). In particular, all zeros of the functions $Ap(z) + Bq(z)$ and $Aq(z) + Bzp(z)$ lie in $(-\infty, 0]$; consequently, they also can be expressed as in (9) due to absence of poles.

If z_0 is such that $F(z_0) = \frac{q(z_0)}{p(z_0)} \leq 0$, then $\phi(z_0; -F(z_0)) = 0$, where we put $\phi(z; A) := Ap(z) + q(z)$. Since for each $A \geq 0$ the function $\phi(z; A)$ does not vanish outside $(-\infty, 0]$, the inequality $z_0 \leq 0$ must be true. Analogously, if z_1 is such that $\frac{F(z_1)}{z_1} = \frac{q(z_1)}{z_1 p(z_1)} \leq 0$, then $\psi(z_1; -F(z_1)/z_1) = 0$, where $\psi(z; B) := Bzp(z) + q(z)$. Since for each $B \geq 0$ the function $\psi(z; B)$ is nonzero outside $(-\infty, 0]$, we obtain $z_1 \leq 0$.

Fact I (Details can be found in e.g. [Du2004, p. 19]). Let $h(z)$ be a real function holomorphic in a neighbourhood of a real point x and such that $h(z) \leq 0$ for a complex z implies $z \leq 0$. Then the expression $h(z) - h(x)$ has a zero at x of some multiplicity $r \geq 1$. Therefore, $h(z) - h(x) \sim (z - x)^r$ as z is close to x in a small enough neighbourhood of x , and we have $\text{Im } h(z) = 0$ on the union of r arcs meeting in this neighbourhood only at x ; one of these arcs is a subinterval of the real line due to the reality of $h(z)$. Furthermore, the half of (if r is even) or all (if r is odd) the arcs contain an interval where $h(z) \leq h(x)$. In particular, the condition $h(x) = 0$ implies $r \leq 2$, and the condition $h(x) < 0$ implies $r = 1$.

Fact I with $h(z) := F(z)$ implies that $F(z)$ can have at most double zeros. It is possible that the function $F(z)$ is holomorphic at the origin and equal to zero there. The assumption that the point $x = 0$ can be a double zero of $F(z)$ is contradictory: Fact I implies that $F(z)$ is negative for all real $z \neq 0$ small enough, which is impossible for $z > 0$. Suppose that $x < 0$ is a double zero of $F(z)$, that is $F(x) = F'(x) = 0 \neq F''(x)$. Then $F(z) < 0$ and, therefore, $z^{-1}F(z) > 0$ for all real z in a sufficiently small punctured neighbourhood of x . At the point x , the function $z^{-1}F(z)$ has a double zero:

$$\frac{F(x)}{x} = \frac{F(x) - xF'(x)}{x^2} = 0 \neq \frac{2x(F(x) - xF'(x)) - x^3F''(x)}{x^4}.$$

Putting $h(z) := z^{-1}F(z)$ in Fact I then yields a contradiction, since the inequality $z^{-1}F(z) \leq 0$ must be satisfied for all real z which are close enough to x . Consequently, the only possible case is $r = 1$, that is that all zeros of $F(z)$ are simple. Considering in the same way $h(z) = \frac{z}{F(z)}$ and $h(z) = \frac{1}{F(z)}$ shows that all poles of $\frac{F(z)}{z}$ are simple. In particular, $F(z)$ cannot have a pole at the origin.

Now, let us prove that zeros and poles of $F(z)$ are interlacing. Suppose that $x_1 < x_2 \leq 0$ are two consecutive zeros of the function $F(z)$, such that the interval (x_1, x_2) contains no poles of $F(z)$. The ratio $z^{-1}F(z)$ also vanishes at x_1 and x_2 unless $x_2 = 0$; therefore, Rolle's theorem gives the points $\xi_1, \xi_2 \in (x_1, x_2)$ such that $F'(\xi_1) = \xi_2^{-2}(F(\xi_2) - \xi_2 F'(\xi_2)) = 0$. Let $h(z) := F(z)$ and $x := \xi_1$ if $F(\xi_1) < 0$, or $h(z) := z^{-1}F(z)$ and $x := \xi_2$ if $F(\xi_1) > 0$. In the special case $x_2 = 0$, the function $F(z)$ is negative in (x_1, x_2) , so we put $h(z) := F(z)$ and denote a zero of $h'(z)$ in this interval by x . Fact I implies $h'(z) \neq 0$ in the whole interval $x_1 < z < x_2 \leq 0$ including $z = x$, which contradicts to our choice of x . This shows that the function $F(z)$ has at least one pole between each pair of its zeros. The same argumentation for $\frac{z}{F(z)}$ instead of $F(z)$ yields that $F(z)$ has a zero between each pair of its poles. As a result, zeros and poles of $F(z)$ are interlacing.

Recall that the functions $p(z)$ and $q(z)$ can be represented as in (9); we can therefore gather all their common zeros and their common exponential term $e^{Az + \frac{A_0}{z}}$ into a function $g_*(z) = \sum_{n=-\infty}^{\infty} g_n z^n \neq 0$ of the form (9), such that zeros of $\frac{q(z)}{g_*(z)}$ coincide with zeros of $F(z)$ and zeros of $\frac{p(z)}{g_*(z)}$ coincide with poles of $F(z)$. In the case $q(z) \equiv 0$, the lemma is trivial. If $(\zeta_1 z + \zeta_2)p(z) = (\eta_1 z + \eta_2)q(z) \neq 0$ with some coefficients $\zeta_1, \zeta_2, \eta_1, \eta_2 \geq 0$, then

$$0 \leq \begin{vmatrix} a_n & a_{n+1} \\ b_n & b_{n+1} \end{vmatrix} = \det \left(\begin{pmatrix} \zeta_2 & \zeta_1 \\ \eta_2 & \eta_1 \end{pmatrix} \cdot \begin{pmatrix} g_n & g_{n+1} \\ g_{n-1} & g_n \end{pmatrix} \right) = \begin{vmatrix} \zeta_2 & \zeta_1 \\ \eta_2 & \eta_1 \end{vmatrix} \cdot \begin{vmatrix} g_n & g_{n+1} \\ g_{n-1} & g_n \end{vmatrix}.$$

Since the Laurent series of $g_*(z)$ converges in any annulus centered at the origin, there exists an integer n such that $g_n^2 > g_{n-1}g_{n+1}$ by Lemma 7. Therefore, the coefficients of the function $F(z) = \frac{\eta_1 z + \eta_2}{\zeta_1 z + \zeta_2}$ satisfy $\zeta_2 \eta_1 \geq \eta_2 \zeta_1$. In the case $\zeta_2 = 0$, we necessarily have $\eta_2 = 0$ due to $q(z) \neq 0$; if $\zeta_2 \neq 0$ then $\alpha_0 = \frac{\zeta_2}{\zeta_1} \geq \frac{\eta_2}{\eta_1} = \beta_0 \geq 0$, which is proving the lemma.

Let the function $F(z)$ be meromorphic and not a constant, and let $C_2(z + \beta_0)p(z) \neq C_1(z + \alpha_0)q(z)$. Since zeros and poles of $F(z)$ are interlacing, this implies that both functions $\frac{q(z)}{g_*(z)}$ and $\frac{p(z)}{g_*(z)}$ have zeros:

$$\frac{p(z)}{g_*(z)} = C_1(z + \alpha_0) \prod_{v>0} \left(1 + \frac{z}{\alpha_v}\right) \quad \text{and} \quad \frac{q(z)}{g_*(z)} = C_2(z + \beta_0) \prod_{v>0} \left(1 + \frac{z}{\beta_v}\right)$$

where $0 \leq \beta_0 < \beta_1 < \dots$ and $0 < \alpha_0 < \alpha_1 < \dots$. For proving that $F(z)$ has the form (14) it is enough to show that the chain inequality $0 < \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \dots$ fails to hold. Let this inequality hold, then $p(z)$ has at least two negative zeros, and thus at least three nonzero coefficients of $p(z)$. By Lemma 6, there are at least three consecutive nonzero coefficients, say a_{i-1}, a_i, a_{i+1} . Estimating the terms according to

$$\frac{1 + \frac{x}{\beta_v}}{1 + \frac{x}{\alpha_v}} < 1 \quad \text{for all } x > 0, \quad v = 1, 2, \dots$$

in the ratio $\frac{q(x) \cdot g_*(x)}{g_*(x) \cdot p(x)}$ yields the contradiction

$$\lim_{x \rightarrow +\infty} F(x) \leq \frac{C_2}{C_1} \cdot \lim_{x \rightarrow +\infty} \frac{x + \beta_0}{x + \alpha_0} = \frac{C_2}{C_1} < \frac{C_2 \beta_0}{C_1 \alpha_0} = F(0) \leq \frac{b_i}{a_i} \leq \lim_{x \rightarrow +\infty} F(x),$$

where the last two inequalities are the first inequality in (12) and the last inequality in (11). Consequently, $\beta_0 < \alpha_0$ and $F(z)$ has the form (14).

Now let us consider the remaining case when the function $F(z)$ has an essential singularity at the origin, that is when the products over $v < 0$ in the representations (9) of $p(z)$ and $q(z)$ have an infinite number of distinct terms. Since the distinct zeros of $p(z)$ and $q(z)$ are interlacing, we can enumerate entries in $(\alpha_v)_{v \neq 0}$ and $(\beta_v)_{v \neq 0}$ so that the inequality (13) is satisfied:

$$\cdots < \alpha_{-2}^{-1} < \beta_{-1}^{-1} < \alpha_{-1}^{-1} < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots,$$

and therefore for all $x > 0$

$$\frac{\alpha_{-1}x}{1 + \alpha_{-1}x} = \frac{1}{1 + \frac{x^{-1}}{\alpha_{-1}}} < \prod_{v=-\infty}^{-1} \frac{1 + \frac{x^{-1}}{\beta_v}}{1 + \frac{x^{-1}}{\alpha_v}} < 1 \quad \text{and} \quad 1 < \prod_{v>0} \frac{1 + \frac{x}{\beta_v}}{1 + \frac{x}{\alpha_v}} < 1 + \frac{x}{\beta_1}.$$

Gathering these estimates together yields

$$\frac{\alpha_{-1}x}{1 + \alpha_{-1}x} < G(x) := \prod_{v=-\infty}^{-1} \frac{1 + \frac{x^{-1}}{\beta_v}}{1 + \frac{x^{-1}}{\alpha_v}} \cdot \prod_{v>0} \frac{1 + \frac{x}{\beta_v}}{1 + \frac{x}{\alpha_v}} < 1 + \frac{x}{\beta_1}; \quad (20)$$

in other words, we put $G(x) = \frac{C_1}{C_2} x^{j-k} F(x)$. Take i is such that $a_{i-2}, a_{i-1}, a_i, a_{i+1} \neq 0$, then $b_{i-1}, b_i, b_{i+1} \neq 0$ by Lemma 8. The series $p(z)$ and $q(z)$ converge to functions holomorphic in $\mathbb{C} \setminus \{0\}$; if $b_i a_{i-1} = b_{i-1} a_i$, then Lemma 9 (b) gives $q(z) = \frac{b_i}{a_i} p(z)$ contradicting to the presence of distinct zeros of $p(z)$ and $q(z)$. Analogously, if $b_i a_i = b_{i+1} a_{i-1}$, then Lemma 13 and Lemma 9 (b) give the contradiction $zp(z) = \frac{a_{i-1}}{b_i} q(z)$. As a result, the inequalities (11) and (12) imply

$$\overline{\lim}_{x \rightarrow 0+} F(x) \leq \frac{b_{i-1}}{a_{i-1}} < \frac{b_i}{a_i} \leq \underline{\lim}_{x \rightarrow +\infty} F(x) \quad \text{and} \quad \underline{\lim}_{x \rightarrow 0+} \frac{F(x)}{x} \geq \frac{b_i}{a_{i-1}} > \frac{b_{i+1}}{a_i} \geq \overline{\lim}_{x \rightarrow +\infty} \frac{F(x)}{x}, \quad (21)$$

According to the latter inequality in (21),

$$\frac{C_2}{C_1} \underline{\lim}_{x \rightarrow 0+} G(x) = \underline{\lim}_{x \rightarrow 0+} x^{j-k} F(x) > \overline{\lim}_{x \rightarrow +\infty} x^{j-k} F(x) = \frac{C_2}{C_1} \overline{\lim}_{x \rightarrow +\infty} G(x)$$

when $j - k \leq -1$ which contradicts to (20). When $j - k \geq 1$, the former inequality in (21) says that

$$\frac{C_2}{C_1} \overline{\lim}_{x \rightarrow 0+} \frac{G(x)}{x} = \overline{\lim}_{x \rightarrow 0+} x^{j-k-1} F(x) < \underline{\lim}_{x \rightarrow +\infty} x^{j-k-1} F(x) = \frac{C_2}{C_1} \underline{\lim}_{x \rightarrow +\infty} \frac{G(x)}{x},$$

which is inconsistent (due to $\alpha_{-1} > \beta_1^{-1}$) with the estimate

$$\alpha_{-1} - \frac{\alpha_{-1}^2 x}{1 + \alpha_{-1} x} < \frac{G(x)}{x} < \frac{1}{\beta_1} + \frac{1}{x}, \quad \text{where } x > 0,$$

equivalent to (20). As a result, the only possible case is $j = k$, that is $F(z)$ coincides with $G(z)$ up to a positive constant and can be expressed as in (13). \square

Lemma 21. *If functions $p(z)$ and $q(z)$ have the form (1) and their ratio $F(z) = \frac{q(z)}{p(z)}$ can be represented as in (13) or (14), then the matrix $H(p, q)$ is totally nonnegative.*

Proof. Indeed, denote by $p_*(z) := \frac{p(z)}{g(z)}$ and $q_*(z) := \frac{q(z)}{g(z)}$ the denominator and numerator of the function $F(z)$ given in (13). This means that $p_*(z) \not\equiv 0$ and $q_*(z)$ have no common zeros, no poles and no exponential factors; the function $g(z)$ has the form (1). The function $C \frac{q_n(z)}{p_n(z)} = F_n(z)$ introduced in (15) maps the upper half-plane into itself for each positive integer n . According to Theorem 3.44 of [HT2012] (see also

Theorem 1.4 of [Dy2014] where the notation is closer to the current paper) the matrix $H(p_n, q_n)$ is totally nonnegative. Since $p_n(z)$ and $q_n(z)$ converge in $\mathbb{C} \setminus \{0\}$ locally uniformly to $p_*(z)$ and $q_*(z)$ respectively, their Laurent coefficients converge as well. Therefore, the matrix $H(p_*, q_*)$ is totally nonnegative as an entry-wise limit of totally nonnegative matrices. Then the Cauchy-Binet formula implies the total nonnegativity of the matrix $H(p, q) = H(p_* g, q_* g) = H(p_*, q_*) \cdot T(g)$, because $T(g)$ is totally nonnegative by Theorem 1. \square

Proof of Theorem 3. Lemma 2 shows that the implications (b) \implies (a) and (a) \implies (b) follow, respectively, from Lemma 20 and Lemma 21. \square

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